Lecture XVI: Applications and Connections

▷ Partition Function of ideal (Non-Interacting) Gas of Quantum Particles

Useful for "normalisation" of interacting theories

e.g. Non-interacting fermions: $\hat{H} = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$

As a warm-up exercise, let us first use coherent state representation:

Quantum Partition function

$$\mathcal{Z}_0 = \text{tr } e^{-\beta(\hat{H}-\mu\hat{N})} = \sum_n \langle n|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle$$

In coherent state basis:

$$\mathcal{Z}_0 = \int d[\bar{\psi},\psi] e^{-\sum_\alpha \bar{\psi}_\alpha\psi_\alpha} \langle -\psi|e^{-\beta(\hat{H}-\mu\hat{N})}|\psi\rangle$$

Using ident. *N.B.* $\hat{n}_\alpha^2 = \hat{n}_\alpha$

$$e^{-\beta(\hat{H}-\mu\hat{N})} = e^{-\beta\sum_\alpha(\epsilon_\alpha-\mu)a_\alpha^\dagger a_\alpha} = \prod_\alpha e^{-\beta(\epsilon_\alpha-\mu)\hat{n}_\alpha} = \prod_\alpha \left[1 + \left(e^{-\beta(\epsilon_\alpha-\mu)} - 1\right)\hat{n}_\alpha\right]$$

$$\begin{aligned}
\mathcal{Z}_0 &= \int d[\bar{\psi},\psi] e^{-\sum_\alpha \bar{\psi}_\alpha\psi_\alpha} \prod_\alpha \left\{ \overbrace{e^{-\bar{\psi}_\alpha\psi_\alpha}}^{\langle -\psi|\psi\rangle} \left[1 + \left(e^{-\beta(\epsilon_\alpha-\mu)} - 1\right)(-\bar{\psi}_\alpha\psi_\alpha)\right] \right\} \\
&= \prod_\alpha \int d\bar{\psi}_\alpha d\psi_\alpha \; \overbrace{e^{-2\bar{\psi}_\alpha\psi_\alpha}}^{1-2\bar{\psi}_\alpha\psi_\alpha} \left[1 + \left(e^{-\beta(\epsilon_\alpha-\mu)} - 1\right)(-\bar{\psi}_\alpha\psi_\alpha)\right] \\
&= \prod_\alpha \int d\bar{\psi}_\alpha d\psi_\alpha \left[1 - 2\bar{\psi}_\alpha\psi_\alpha - \left(e^{-\beta(\epsilon_\alpha-\mu)} - 1\right)\bar{\psi}_\alpha\psi_\alpha\right] \\
&= \prod_\alpha \int d\bar{\psi}_\alpha d\psi_\alpha \left[-\bar{\psi}_\alpha\psi_\alpha(1 + e^{-\beta(\epsilon_\alpha-\mu)})\right] \\
&= \prod_\alpha \left[1 + e^{-\beta(\epsilon_\alpha-\mu)}\right] \qquad \text{i.e. Fermi} - \text{Dirac distribution}
\end{aligned}$$

Exercise: show (using CS) that in Bosonic case

$$\mathcal{Z}_0 = \prod_\alpha \sum_{n=0}^{\infty} e^{-n\beta(\epsilon_\alpha-\mu)} = \prod_\alpha \left[1 - e^{-\beta(\epsilon_\alpha-\mu)}\right]^{-1}$$

Bose-Einstein distribution

▷ Connection of CSPI with FPI

e.g. Quantum Harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2$

In second quantised form, $\hat{H} = (a^\dagger a + 1/2)\hbar\omega$, $\qquad [a, a^\dagger] = 1$, i.e. bosons!

$$\mathcal{Z} = \mathrm{tr}(e^{-\beta\hat{H}}) = \int_{\psi(\beta)=\psi(0)}^{\bar{\psi}(\beta)=\bar{\psi}(0)} D[\bar{\psi},\psi] \exp\left[-\int_0^\beta \left(\bar{\psi}\partial_\tau\psi + \hbar\omega\bar{\psi}\psi\right)\right]$$

$e^{-\beta\hbar\omega/2}$ *in* $D[\bar{\psi},\psi]$, $\psi(\tau)$ — complex scalar field

Parameterise complex field in terms of two real scalar fields

$$\psi(\tau) = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left[q(\tau) + \frac{ip(\tau)}{m\omega}\right]$$

Substituting (e.g. $\hbar\omega\bar{\psi}\psi = \frac{m\omega^2}{2}(q^2 + \frac{p^2}{(m\omega)^2})$) and noting $\int_0^\beta d\tau\, q\dot{p} = -\int_0^\beta d\tau\, p\dot{q}$

$$\mathcal{Z} = \int_{\psi(\beta)=\psi(0)}^{\bar{\psi}(\beta)=\bar{\psi}(0)} D[p,q] \exp\left[-\int_0^\beta d\tau \left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 - \frac{ip\dot{q}}{\hbar}\right)\right]$$

cf. (Euclidean time) FPI $\beta = it/\hbar$, $\tau = it'/\hbar$, $\frac{i}{\hbar}\frac{\partial q}{\partial\tau} = \frac{\partial q}{\partial t'}$

$$\mathcal{Z} = \int D[p,q] \exp\left[\frac{i}{\hbar}\int_0^t dt'\left(p\dot{q} - H(p,q)\right)\right]$$

Partition Function of Harmonic Oscillator from CSPI

(i) Bosonic oscillator:

$$\mathcal{Z}_B = \overbrace{\int D[\bar{\psi},\psi] \exp\left[-\int_0^\beta d\tau\bar{\psi}\left(\partial_\tau + \hbar\omega\right)\psi\right]}^{J\det(\partial_\tau+\hbar\omega)^{-1}} = \int (\prod_n d\bar{\psi}_{\omega_n} d\psi_{\omega_n}) e^{-\sum_n \bar{\psi}_{\omega_n}(i\omega_n + \hbar\omega)\psi_{\omega_n}}$$

$$= J\prod_{\omega_n}[i\omega_n + \hbar\omega]^{-1} = \frac{J}{\hbar\omega}\prod_{n=1}^\infty\left[(\hbar\omega)^2 + \left(\frac{2n\pi}{\beta}\right)^2\right]^{-1} \frac{J'}{\hbar\omega}\prod_{n=1}^\infty\left[1 + \left(\frac{\hbar\omega\beta}{2\pi n}\right)^2\right]^{-1}$$

$$= \frac{J'}{2\beta\sinh(\hbar\omega\beta/2)}$$

$\prod_{n=1}^\infty[1 + (x/\pi n)^2] = (\sinh x)/x$

Normalisation: $T \to 0$, $\mathcal{Z}$ dominated by g.s. $\lim_{\beta\to\infty}\mathcal{Z}_B = e^{-\beta\hbar\omega/2}(= \mathcal{Z}_F)$

$$\text{i.e. } J' = \beta \qquad \mathcal{Z}_B = \frac{1}{2\sinh(\hbar\beta\omega/2)}$$

(ii) Fermionic oscillator: Gaussian Grassmann integration

$$\begin{aligned}
\mathcal{Z}_F &= J\det(\partial_\tau + \hbar\omega) = J\prod_{\omega_n}[i\omega_n + \hbar\omega] = J\prod_{n=0}^{\infty}\left[(\hbar\omega)^2 + \left(\frac{(2n+1)\pi}{\beta}\right)^2\right] \\
&= J'\prod_{n=1}^{\infty}\left[1 + \left(\frac{\hbar\omega\beta}{(2n+1)\pi}\right)^2\right] = J'\cosh(\hbar\omega\beta/2)
\end{aligned}$$

$\prod_{n=1}^{\infty}[1 + (x/\pi(2n+1))^2] = \cosh(x/2)$

Using normalisation: $\lim_{\beta\to\infty}\mathcal{Z}_F = e^{-\beta\hbar\omega/2}$

$$J' = 2e^{-\beta\hbar\omega} \qquad \mathcal{Z}_F = 2e^{-\beta\hbar\omega}\cosh(\hbar\beta\omega/2).$$

cf. direct computation:

$$\mathcal{Z}_B = e^{-\beta\hbar\omega/2}\sum_{n=0}^{\infty}e^{-n\beta\hbar\omega}, \qquad \mathcal{Z}_F = e^{-\beta\hbar\omega/2}\sum_{n=0}^{1}e^{-n\beta\hbar\omega}.$$

Note that normalising prefactor J' involves only a constant offset of free energy,

$$F = -k_{\mathrm{B}}T\ln\mathcal{Z}$$

statistical correlations encoded in content of functional integral

▷ *In notes, two case studies:*

(i) Plasma Theory of the weakly interacting electron gas

(ii) BCS theory of superconductivity — a prototype for gauge theories

We will deal with project (ii)